CLASSIFICATION OF THREE-DIMENSIONAL ZEROPOTENT ALGEBRAS OVER AN ALGEBRAICALLY CLOSED FIELD

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Abstract. A nonassociative algebra is defined to be zeropotent if the square of any element is zero. Zeropotent algebras are exactly the same as anticommutative algebras when the characteristic of the ground field is not two. Also, in general, the class of zeropotent algebras properly contains that of Lie algebras. In this paper, we give a complete classification of three-dimensional zeropotent algebras over an algebraically closed field of characteristic not equal to two up to isomorphism. By restricting the result to the subclass of Lie algebras, we can obtain a classification of three-dimensional complex Lie algebras, which is in accordance with the conventional one.

1. Introduction

Let K be a field and char K denote the characteristic of K. A nonassociative algebra A over K is defined to be *zeropotent*[*1] if $x^2 = 0$ for all $x \in A$. A zeropotent algebra A is *anticommutative*, that is, $xy = -yx$ for all $x, y \in A$, and the converse is true if char K is not two. As is well-known, a zeropotent algebra A is said to be a *Lie algebra* if it further satisfies the Jacobi identity $(xy)z + (yz)x + (zx)y = 0$ for all $x, y, z \in A$. In general, the class of zeropotent algebras properly contains that of Lie algebras. For a systematic exposition of nonassociative algebras, we refer the reader to [6]. For more details on Lie algebras, see [3].

Obviously, a one-dimensional zeropotent algebra is the unique algebra Kx with $x^2 = 0$, which is an abelian Lie algebra. Also, it is straightforward that a two-dimensional zeropotent algebra $Kx + Ky$ has two isomorphism classes defined by (1) $xy = 0$ (an abelian Lie algebra) and by (2) $xy = x$ (a Lie algebra which is non-abelian if char $K \neq 2$).

The aim of this paper is to completely classify three-dimensional zeropotent algebras over an algebraically closed filed K of char $K \neq 2$ up to isomorphism. Let us define the equivalence relation $\sim$ in K by $a \sim b$ if and only if $a = \pm b$ for $a, b \in K$. Then let $\mathcal{H} \subset K$ be a complete set of equivalence class representatives for $K/\sim$. Typically, in the case where K is the complex number field $\mathbb{C}$, $\mathcal{H}$ is taken to be the half plane $\{z \in \mathbb{C} \mid -\pi/2 < \arg(z) \leq \pi/2\} \cup \{0\}$. Our result is that the algebras are classified into ten families

$$A_0, A_1, A_2, A_3, \{A_4(a)\}_{a\in\mathcal{H}}, A_5, A_6, \{A_7(a)\}_{a\in\mathcal{H}}, A_8 \text{ and } A_9$$

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[*1] The terminology "zeropotent" is also used for the more general algebraic structure *groupoid* in [4].

up to isomorphism, where A_i is a type of the *structure matrix* with respect to a linear base whose entries are determined from the product between each pair of the base. Two of the ten families are parametrized by a in $\mathcal{H}$, which implies that there exist an infinite number of non-isomorphic algebras in these families. The details will be described in Theorem 8.1.

By restricting the result to the subclass of Lie algebras over $\mathbb{C}$, we can easily obtain a classification of three-dimensional complex Lie algebras up to isomorphism. This classification is in accordance with the conventional one as found in [3, 2]. This will be briefly mentioned as a final remark.

The rest of this paper is organized as follows. In Section 2, we characterize three-dimensional zeropotent algebras by the above mentioned structure matrices. In the term of an equivalence relation between the matrices, we give a criterion for isomorphism between three-dimensional zeropotent algebras (Proposition 2.1). By this, the problem of classifying three-dimensional zeropotent algebras comes down to that of determining equivalence classes of structure matrices, which is a central theme of this paper. In Section 3, we give a necessary and sufficient condition for a zeropotent algebra to be a Lie algebra (Proposition 3.2). We call a zeropotent algebra *symmetric* if its structure matrix is symmetric, and give a close relationship between symmetric algebras and Lie algebras (Corollary 3.3). Other than rank and symmetry, we propose a new 'invariant' called the *jacobi element*, which will play a crucial role in proving non-isomorphism of two different types of zeropotent algebras (Proposition 3.4). We separate zeropotent algebras into two categories: *curly* and *straight*. That is, a zeropotent algebra is curly if the product of any two elements comes into the space spanned by these elements, otherwise it is straight. Section 4 completely determines three-dimensional curly algebras (Proposition 4.1). The subsequent three sections concern straight algebras when K is an algebraically closed field of char $K \neq 2$. In Section 5, we show that the structure matrix of any straight algebra is equivalent to some triangular matrix (Lemma 5.2), and present a canonical form of the triangular matrix for each rank (Corollary 5.3). Based on this triangulation, we shall classify three-dimensional straight algebras for ranks 1 and 2 in Section 6 (Propositions 6.1 and 6.3) and for rank 3 in Section 7 (Proposition 7.2). Finally, Section 8 completes the desired classification by summarizing the above results and gives some concluding remarks.

2. A criterion for isomorphism

In this and the next two sections K is an arbitrary field. Let A be a zeropotent algbera over K of dimension 3. Let $\{e_1, e_2, e_3\}$ be a linear base of A. Because A is zeropotent, $e_1^2 = e_2^2 = e_3^2 = 0, e_1e_2 = -e_2e_1, e_1e_3 = -e_3e_1$ and $e_2e_3 = -e_3e_2$. Write

$$\begin{cases} e_2e_3 = a_{11}e_1 + a_{12}e_2 + a_{13}e_3 \\ e_3e_1 = a_{21}e_1 + a_{22}e_2 + a_{23}e_3 \\ e_1e_2 = a_{31}e_1 + a_{32}e_2 + a_{33}e_3 \end{cases} \tag{1}$$

with $a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, a_{23}, a_{31}, a_{32}, a_{33} \in K$. We can rewrite (1) as

$$\begin{pmatrix} e_2e_3 \\ e_3e_1 \\ e_1e_2 \end{pmatrix} = A \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} \tag{2}$$

with the matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix},$$

which is called the *structure matrix* with respect to the base $\{e_1, e_2, e_3\}$. The structure matrix is determined by the algebra, and vice versa. Hence, we hereafter will say that the algebra is defined by the structure matrix, and freely use the same symbol A both for the matrix and for the algebra if there is no confusion.

Let A' be another zeropotent algebra on a base $\{e'_1, e'_2, e'_3\}$ defined by

$$\begin{pmatrix} e'_2e'_3 \\ e'_3e'_1 \\ e'_1e'_2 \end{pmatrix} = A' \begin{pmatrix} e'_1 \\ e'_2 \\ e'_3 \end{pmatrix}, \tag{3}$$

where

$$A' = \begin{pmatrix} a'_{11} & a'_{12} & a'_{13} \\ a'_{21} & a'_{22} & a'_{23} \\ a'_{31} & a'_{32} & a'_{33} \end{pmatrix}$$

with $a'_{11}, a'_{12}, a'_{13}, a'_{21}, a'_{22}, a'_{23}, a'_{31}, a'_{32}, a'_{33} \in K$. That is, A' is the structure matrix with respect to the base $\{e'_1, e'_2, e'_3\}$.

Suppose that A and A' are isomorphic, and let $\Phi : A \to A'$ be an isomorphism. Let

$$X = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix} \tag{4}$$

with $x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33} \in K$ be the matrix associated with the linear map Φ, that is,

$$\begin{pmatrix} \Phi(e_1) \\ \Phi(e_2) \\ \Phi(e_3) \end{pmatrix} = X \begin{pmatrix} e'_1 \\ e'_2 \\ e'_3 \end{pmatrix}. \tag{5}$$

Since Φ is an isomorphism, we have

$$\begin{pmatrix} \Phi(e_2)\Phi(e_3) \\ \Phi(e_3)\Phi(e_1) \\ \Phi(e_1)\Phi(e_2) \end{pmatrix} = \begin{pmatrix} \Phi(e_2e_3) \\ \Phi(e_3e_1) \\ \Phi(e_1e_2) \end{pmatrix} = A \begin{pmatrix} \Phi(e_1) \\ \Phi(e_2) \\ \Phi(e_3) \end{pmatrix} = AX \begin{pmatrix} e'_1 \\ e'_2 \\ e'_3 \end{pmatrix}, \tag{6}$$

using (2) and (5). On the other hand we have

$$\begin{aligned} \Phi(e_2)\Phi(e_3) &= (x_{21}e'_1 + x_{22}e'_2 + x_{23}e'_3)(x_{31}e'_1 + x_{32}e'_2 + x_{33}e'_3) \\ &= (x_{22}x_{33} - x_{23}x_{32})e'_2e'_3 - (x_{21}x_{33} - x_{23}x_{31})e'_3e'_1 + (x_{21}x_{32} - x_{22}x_{31})e'_1e'_2, \end{aligned}$$

$$\begin{aligned} \Phi(e_3)\Phi(e_1) &= (x_{31}e'_1 + x_{32}e'_2 + x_{33}e'_3)(x_{11}e'_1 + x_{12}e'_2 + x_{13}e'_3) \\ &= -(x_{12}x_{33} - x_{13}x_{32})e'_2e'_3 + (x_{11}x_{33} - x_{13}x_{31})e'_3e'_1 - (x_{11}x_{32} - x_{12}x_{31})e'_1e'_2 \end{aligned}$$

and

$$\begin{aligned}\Phi(e_1)\Phi(e_2) &= (x_{11}e'_1 + x_{12}e'_2 + x_{13}e'_3)(x_{21}e'_1 + x_{22}e'_2 + x_{23}e'_3) \\ &= (x_{12}x_{23} - x_{13}x_{22})e'_2e'_3 - (x_{11}x_{23} - x_{13}x_{21})e'_3e'_1 + (x_{11}x_{22} - x_{12}x_{21})e'_1e'_2.\end{aligned}$$

Hence,

$$\begin{pmatrix}\Phi(e_2)\Phi(e_3)\\ \Phi(e_3)\Phi(e_1)\\ \Phi(e_1)\Phi(e_2)\end{pmatrix} = Y \begin{pmatrix}e'_2e'_3\\ e'_3e'_1\\ e'_1e'_2\end{pmatrix} = YA' \begin{pmatrix}e'_1\\ e'_2\\ e'_3\end{pmatrix} \tag{7}$$

by (3), where

$$Y = \begin{pmatrix}y_{11} & y_{12} & y_{13}\\ y_{21} & y_{22} & y_{23}\\ y_{31} & y_{32} & y_{33}\end{pmatrix}$$

and y_{ij} is the (ij)-cofactor of X. Because $Y = |X|\,{}^tX^{-1}$ by (6) and (7), we get

$$A' = \frac{1}{|X|}\,{}^tXAX. \tag{8}$$

Conversely, if (8) holds, the linear map associated with X is an isomorphism. Thus, we have

Proposition 2.1. *Let A and A' be three-dimensional zeropotent algebras over K. Then, A and A' are isomorphic if and only if there is a nonsingular matrix X satisfying (8).*

We remark that the equality (8) also appears in [2, 5] for Lie algebras, but our proposition is claimed for the wider class of zeropotent algebras.

Corollary 2.2. *If A and A' are isomorphic, then rank A=rank A'.*

Corollary 2.3. *If $|A| = |A'| \neq 0$, then A and A' are isomorphic if and only if there is a matrix X such that $|X| = 1$ and*

$$A' = {}^tXAX. \tag{9}$$

Proof. If (8) holds, then

$$|A'| = \frac{1}{|X|^3}\,|{}^tX|\,|A|\,|X| = \frac{|A|}{|X|}.$$

Therefore, if $|A| = |A'| \neq 0$, then $|X| = 1$ and (9) holds. □

Corollary 2.4. *If K is algebraically closed, then A and A' are isomorphic if and only if there is a nonsingular matrix X satisfying (9).*

Proof. Replacing X by $\frac{X}{\sqrt{|X|}}$ in (8), we get (9). Conversely, replacing X by $\frac{X}{|X|}$ in (9), we get (8). □

When (8) holds, we say that the matrices A and A' are *equivalent* and refer to X as a *transformation matrix* for the equivalence $A \cong A'$. Also, when using the symbols A and A' as algebras, we call this X a transformation matrix for the isomorphism $A \cong A'$ as well.

3. Symmetric algebras and Lie algebras

Let A be a zeropotent algebra of dimension 3 with base $\{e_1, e_2, e_3\}$ over K. We say that A is a *symmetric algebra* if the matrix A defining it is symmetric. Symmetry is an invariant property of algebras. In fact,

Proposition 3.1. *If A is symmetric and A is isomorphic to A', then A' is symmetric.*

Proof. By Proposition 2.1, $A' = \frac{1}{|X|}\,{}^tXAX$ for a nonsingular matrix X. Hence, if A is symmetric, then so is A'. ☐

For $\alpha, \beta, \gamma \in K$, the diagonal matrix $\begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix}$ is denoted by $D(\alpha, \beta, \gamma)$. Algebras defined by diagonal matrices are typical symmetric algebras.

We define the *jacobi element* $\text{jac}(A)$ of A (with respect to the base $\{e_1, e_2, e_3\}$) by

$$\text{jac}(A) = (e_1, e_2, e_3) \begin{pmatrix} e_2e_3 \\ e_3e_1 \\ e_1e_2 \end{pmatrix}.$$

By (1) we have

$$\text{jac}(A) = (e_1, e_2, e_3)A\,{}^t(e_1, e_2, e_3) = b_1e_2e_3 + b_2e_3e_1 + b_3e_1e_2 = (b_1, b_2, b_3)A\,{}^t(e_1, e_2, e_3), \tag{10}$$

where $b_1 = a_{23} - a_{32}, b_2 = a_{31} - a_{13}$ and $b_3 = a_{12} - a_{21}$.

Proposition 3.2. *A is a Lie algebra if and only if*

$$(b_1, b_2, b_3)A = (0, 0, 0). \tag{11}$$

Proof. The Jacobi identity $\text{jac}(A) = 0$ holds if and only if the right hand side in (10) is equal to 0, if and only if (11) holds. ☐

Corollary 3.3. *A symmetric algebra is a Lie algebra. When rank $A = 3$, then A is a Lie algebra if and only if A is symmetric.*

Proof. If A is symmetric, then $b_1 = b_2 = b_3 = 0$ and so (11) holds. If A is nonsingular, then (11) holds, if and only if $b_1 = b_2 = b_3 = 0$, if and only if A is symmetric. ☐

Let A' be another algebra on a base $\{e'_1, e'_2, e'_3\}$ defined by a matrix A' and let $\Phi : A \to A'$ be an isomorphism with the associated matrix X in (4). Then, by (5) and (6) we have

$$\Phi(\text{jac}(A)) = (\Phi(e_1), \Phi(e_2), \Phi(e_3)) \begin{pmatrix} \Phi(e_2e_3) \\ \Phi(e_3e_1) \\ \Phi(e_1e_2) \end{pmatrix} = (e'_1, e'_2, e'_3)\,{}^tXAX \begin{pmatrix} e'_1 \\ e'_2 \\ e'_3 \end{pmatrix}.$$

The last term is equal to

$$|X|(e'_1, e'_2, e'_3)A'\,{}^t(e'_1, e'_2, e'_3) = |X|\text{jac}(A')$$

by (8). Hence we have

Proposition 3.4. *If A and A' are isomorphic with a transformation matrix X, and if* $\mathrm{jac}(A) = a_1e_1 + a_2e_2 + a_3e_3$ *and* $\mathrm{jac}(A') = a'_1e'_1 + a'_2e'_2 + a'_3e'_3$, *then we have*

$$(a_1, a_2, a_3)X = |X|(a'_1, a'_2, a'_3).$$

This proposition claims that the jacobi element is in a sense an invariant of algebras, which will play a crucial role in proving non-isomorphism of two different types of zeropotent algebras in Sections 6 and 7.

4. Curly algebras

A zeropotent algebra A is *curly* if for any $e, f \in A$, their product ef is in the space spanned by $\{e, f\}$, otherwise A is *straight*. We call a curly zeropotent algebra and a straight zeropotent algebra simply a *curly algebra* and a *straight algebra*, respectively. In this section, we assume that A is a curly algebra of dimension 3 over K.

Let $\{e, f, g\}$ be a linear base of A. Since A is curly, $fg = af + bg, ge = a'e + cg$ and $ef = b'e + c'f$ for $a, b, c, a', b', c' \in K$. Because

$$(e + f)g = eg + fg = -a'e + af + (b - c)g$$

is in $K\{e+f, g\}$, we see $a' = -a$. Similarly, we have $b' = -b$ and $c' = -c$. Hence A is defined by

$$A(a, b, c) = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}.$$

First, let $A_0 = A(0, 0, 0)$. A_0 is obviously a curly algebra which is an abelian Lie algebra.

If $a \neq 0$, let $X = \begin{pmatrix} 1 & 0 & c \\ 0 & 1 & -b \\ 0 & 0 & a \end{pmatrix}$. Then, $|X| = a$ and

$$\frac{1}{|X|}\, {}^tXAX = A(1, 0, 0). \tag{12}$$

If $c \neq 0$, let $X = \begin{pmatrix} c & 0 & 0 \\ -b & 1 & 0 \\ a & 0 & 1 \end{pmatrix}$. Then, $|X| = c$ and

$$\frac{1}{|X|}\, {}^tXAX = A(0, 0, 1). \tag{13}$$

If $b \neq 0$, let $X = \begin{pmatrix} 1 & c & 0 \\ 0 & -b & 0 \\ 0 & a & 1 \end{pmatrix}$. Then, $|X| = b$ and

$$\frac{1}{|X|}\, {}^tXAX = A(0, 1, 0). \tag{14}$$

Thus, we see that the algebra defined by $A(1, 1, 1)$ is isomorphic to these algebras defined by the right hand sides of (12), (13) and (14). Consequently, all these algebras are isomorphic to each other. Let $A_3 = A(1, 0, 0)$. Apparently A_0 is not isomorphic to A_3 since rank $A_0 \neq$ rank A_3. In conclusion, we have

Proposition 4.1. *Up to isomorphism, there are exactly two curly algebras defined by*

$$A_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } A_3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

5. Straight algebras and triangulation

From this section K is an algebraically closed field of characteristic not equal to 2. Let A be a straight algebra of dimension 3 over K. Let $\{e, f, ef\}$ be a linear base of A such that $g = ef$. Then, A is defined by

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & 1 \end{pmatrix} \tag{15}$$

for $a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, a_{23} \in K$.

A 2×2 matrix

$$B = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \tag{16}$$

with $a_{11}, a_{12}, a_{21}, a_{22} \in K$ is *marginal* if

$$a_{11} = a_{22} = a_{12} + a_{21} = 0.$$

In this case we also say that A is marginal. We start with the following easy lemma. For $\alpha, \beta \in K$, let $D(\alpha, \beta)$ denote the diagonal matrix $\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$.

Lemma 5.1. *If B in (16) is not marginal, then there exists a nonsingular matrix Y over K such that*

$${}^tYBY = \begin{pmatrix} \varepsilon_1 & a \\ 0 & \varepsilon_2 \end{pmatrix}$$

with $a \in K$ and $\varepsilon_1, \varepsilon_2 \in \{0, 1\}$.

Proof. If $a_{21} = 0$, then let $X = D(1, 1)$. If $a_{22} \neq 0$, let $X = \begin{pmatrix} 1 & 0 \\ -a_{21}/a_{22} & 1 \end{pmatrix}$. If $a_{21} \neq 0, a_{22} = 0$ and $[a_{11} \neq 0$ or $a_{12} + a_{21} \neq 0]$, then taking $d \in K$ so that $a_{11} + (a_{12} + a_{21})d \neq 0$, let $X = \begin{pmatrix} 1 & 1 \\ d & -(a_{11} + a_{12}d)/a_{21} \end{pmatrix}$. In any case, $|X| \neq 0$ and ${}^tXBX$ is of the form $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$. Otherwise, B is marginal.

Let $\alpha = \sqrt{a}$ if $a \neq 0$, and let $\alpha = 1$ if $a = 0$. Let $\gamma = \sqrt{c}$ if $c \neq 0$, and let $\gamma = 1$ if $c = 0$. Now set $Y = XD(1/\alpha, 1/\gamma)$, then ${}^tYBY = \begin{pmatrix} \varepsilon_1 & b/\alpha\gamma \\ 0 & \varepsilon_2 \end{pmatrix}$ with $\varepsilon_1, \varepsilon_2 \in \{0, 1\}$. □

For $a, b, c \in K$ and $\varepsilon_1, \varepsilon_2 \in \{0, 1\}$, let

$$T(a, b, c, \varepsilon_1, \varepsilon_2) = \begin{pmatrix} \varepsilon_1 & a & b \\ 0 & \varepsilon_2 & c \\ 0 & 0 & 1 \end{pmatrix}.$$

Lemma 5.2. *For any matrix A in (15), there exists a nonsingular matrix X such that*

$$ {}^tXAX = T(a,b,c,\varepsilon_1,\varepsilon_2) $$

with $a,b,c \in K$ and $\varepsilon_1,\varepsilon_2 \in \{0,1\}$.

Proof. First, suppose that $B = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ is marginal, that is, $a_{11} = a_{22} = a_{12} + a_{21} = 0$. Let $a = a_{12}, b = a_{13}$ and $c = a_{23}$. If $a = 0$, then $a_{21} = -a = 0$ and $A = T(0,b,c,0,0)$. Thus, we may suppose that $a \neq 0$. Let

$$ X_1 = \begin{pmatrix} (c+d)/2a & 0 & 0 \\ 0 & a & 1 \\ 1 & 0 & (-c+d)/2 \end{pmatrix} {}^{*2}, $$

where $d = \sqrt{c^2+4}$, then we have

$$ {}^tX_1AX_1 = \begin{pmatrix} 1 + b(c+d)/2a & a(c+d)/2 & (b+ad)/a \\ a(c-d)/2 & 0 & ac(d-c)/2 \\ 0 & 0 & 1 \end{pmatrix}. $$

This matrix is not marginal if $c \neq 0$ or $a + b \neq 0$. If $c = a + b = 0$, let $X_2 = \begin{pmatrix} 0 & 1/a & 0 \\ -a & 1 & 2 \\ 0 & 2 & 1 \end{pmatrix}$.

Then, we have ${}^tX_2AX_2 = \begin{pmatrix} 0 & a & 0 \\ -a & 2 & 3 \\ 0 & 0 & 1 \end{pmatrix}$ and this matrix is not marginal either.

If B is not marginal, then by Lemma 5.1, there is a nonsingular 2×2 matrix Y such that ${}^tYBY = \begin{pmatrix} \varepsilon_1 & a \\ 0 & \varepsilon_2 \end{pmatrix}$ with $a \in K$ and $\varepsilon_1, \varepsilon_2 \in \{0,1\}$. Let $X = \begin{pmatrix} Y & 0 \\ 0 & 1 \end{pmatrix}$, then ${}^tXAX = T(a,b,c,\varepsilon_1,\varepsilon_2)$ for some $b,c \in K$. □

Corollary 5.3. *Let A be a straight algebra. If rank $A = 1$, then A can be defined by $T(0,b,c,0,0)$ with $b,c \in K$. If rank $A = 2$, then A can be defined by $T(a,b,c,0,1)$ with $a,b,c \in K$. If rank $A = 3$, then A can be defined by $T(a,b,c,1,1)$ with $a,b,c \in K$.*

Proof. Clearly rank $T(a,b,c,\varepsilon_1,\varepsilon_2) = 1$ if and only if $a = \varepsilon_1 = \varepsilon_2 = 0$, and rank $T(a,b,c,\varepsilon_1,\varepsilon_2) = 3$ if and only if $\varepsilon_1 = \varepsilon_2 = 1$. When rank $T(a,b,c,\varepsilon_1,\varepsilon_2) = 2$, we have three cases where (i) $\varepsilon_1 = 0, \varepsilon_2 = 1$, (ii) $\varepsilon_1 = 1, \varepsilon_2 = 0$ and (iii) $\varepsilon_1 = \varepsilon_2 = 0, a \neq 0$.

Put

$$ X_0 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, $$

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*2 We found such a matrix by solving the system of nine algebraic equations in nine variables derived from a given matrix equation ${}^tXAX = A'$. In general, however, this task is very hard and so we often used computational algebraic techniques including *Gröbner basis*. This is not the subject of the present paper and we will not go into the details. Most transformation matrices that appear in this paper were computed in this manner. In some of the computations, we used computer algebra system *Mathematica* or *Maple*, especially for Gröbner basis computation and linear algebra operations. However, we finally checked that every obtained transformation matrix satisfies the associated matrix equation by *hand calculations*.

and for $a \in K \setminus \{0\}$ let

$$X_1 = \begin{pmatrix} 1 & 1 & 0 \\ -1/a & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } X_2 = \begin{pmatrix} 1 & 1/a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then, we see

$${}^tX_0T(0,b,c,1,0)X_0 = T(0,-c,b,0,1),$$

and if $a \neq 0$, we see

$${}^tX_1T(a,b,c,1,0)X_1 = T(1,b-c/a,b,0,1)$$

and

$${}^tX_2T(a,b,c,0,0)X_2 = T(a,b,c+b/a,0,1).$$

Hence, any matrix in the cases (ii) and (iii) is equivalent to a matrix in (i), and so A can be defined by $T(a,b,c,0,1)$ for some $a,b,c \in K$. □

6. Straight algebras of rank 1 and rank 2

Let A be a straight algebra of rank 1. By Corollary 5.3, A can be defined by

$$B(a,b) = T(0,a,b,0,0) = \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & b \\ 0 & 0 & 1 \end{pmatrix}$$

with $a,b \in K$. If $b \neq 0$, let $X = \begin{pmatrix} 0 & 1 & 0 \\ 1/b & -a/b & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Then we have

$$ {}^tXB(a,b)X = B(1,0). \tag{17}$$

If $a \neq 0$, we have

$$D(1/a,1,1)B(a,b)D(1/a,1,1) = B(1,b). \tag{18}$$

Letting $A_1 = B(0,0)$ and $A_2 = B(1,0)$, we have

Proposition 6.1. *Up to isomorphism, there are exactly two straight algebras of rank 1 defined by*

$$A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \textit{ and } A_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Proof. If $(a,b) \neq (0,0)$, then $B(a,b)$ is isomorphic to A_2 by (17) and (18), otherwise A is isomorphic to A_1. Note that A_1 is symmetric but A_2 is not, hence they are not isomorphic. □

Next, we shall study straight algebras of rank 2. Let

$$C(a,b,c) = T(a,b,c,0,1) = \begin{pmatrix} 0 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$$

with $a, b, c \in K$. Any straight algebra A of rank 2 is defined by $C(a, b, c)$ by Corollary 5.3. Let A_5 be the algebra defined by $T(1, 0, 0, 0, 0)$. Let

$$X_1 = \begin{pmatrix} d/h & -1/d & c/\sqrt{h} \\ 0 & a/d & -b/\sqrt{h} \\ 0 & 1 & a/\sqrt{h} \end{pmatrix},$$

where $d = b - ac$ and $h = a^2 + b^2 - abc$. If $d \neq 0$ and $h \neq 0$, we have that $|X_1| = 1/\sqrt{h} \neq 0$ and

$$^tX_1 C(a, b, c) X_1 = T(1, 0, 0, 0, 0).$$

Hence, (i) if $b - ac \neq 0$ and $a^2 + b^2 - abc \neq 0$, A is isomorphic to A_5. Let

$$X_2 = \begin{pmatrix} 1/a & 0 & c/a \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{pmatrix}.$$

If $a \neq 0$, we have $|X_2| = 1/a \neq 0$ and

$$^tX_2 C(a, ac, c) X_2 = T(1, 0, 0, 0, 0).$$

Thus, (ii) if $b = ac$ and $a \neq 0$, A is again isomorphic to A_5. Let A_6 be the algebra defined by $T(1, 1, 1, 0, 0)$, and let

$$X_3 = \begin{pmatrix} 1/b & -1/b & (b^2 - a^2)/a^2 b \\ 0 & 0 & -b/a \\ 0 & 1 & 2 \end{pmatrix}.$$

If $ab \neq 0$, then $|X_3| = 1/a \neq 0$ and

$$^tX_3 C(a, b, (a^2 + b^2)/ab) X_3 = T(1, 1, 1, 0, 0).$$

Hence, (iii) if $a^2 + b^2 - abc = 0, a \neq 0$ and $b \neq 0$, then A is isomorphic to A_6. As the other case than (i)-(iii) above, we have the case (iv) $a = b = 0$, and then A has become $C(0, 0, c)$. Thus, let $A_4(a)$ with $a \in K$ denote the algebra defined by $C(0, 0, a)$.

Lemma 6.2.

(1) *For* $a, b \in K$, $A_4(a) \cong A_4(b)$ *if and only if* $a = \pm b$.

(2) *For all* $a \in K$, $A_5 \ncong A_4(a)$ *and* $A_6 \ncong A_4(a)$.

(3) $A_5 \ncong A_6$.

Proof. (1) $A_4(a)$ and $A_4(-a)$ are isomorphic, because $^tX_0 A_4(a) X_0 = A_4(-a)$ with $X_0 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & a \end{pmatrix}$. On the other hand, $A_4(0)$ is not isomorphic to $A_4(a)$ with $a \in K \setminus \{0\}$ because $A_4(0)$ is symmetric but $A_4(a)$ is not.

Suppose that $a \neq 0, b \neq 0$ and $A_4(a)$ is isomorphic to $A_4(b)$. Then, by Corollary 2.4 there is a matrix $X = (x_{ij})$ satisfying

$$^tX A_4(a) X = A_4(b). \tag{19}$$

Let $X_1 = \begin{pmatrix} x_{22} & x_{23} \\ x_{32} & x_{33} \end{pmatrix}$, and $B(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$. Then, by (19) we have

$$^tX_1 B(a) X_1 = B(b). \tag{20}$$

Because $|X_1|^2|B(a)| = |B(b)|$ and $|B(a)| = |B(b)| = 1$, we see $|X_1| = \pm 1$. Let $Y = \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix}$ be the matrix in the left hand side of (20). Then, we have

$$\begin{cases} y_{12} = x_{22}x_{23} + (ax_{22} + x_{32})x_{33} = b \\ y_{21} = x_{22}x_{23} + x_{32}(ax_{23} + x_{33}) = 0. \end{cases}$$

Hence,

$$b = y_{12} - y_{21} = a(x_{22}x_{33} - x_{23}x_{32}).$$

Therefore, $b/a = |X_1| = \pm 1$, and hence $a = \pm b$.

(2) Note that by Proposition 3.2 $A_4(a)$ is a Lie algebra for any $a \in K$ but A_5 and A_6 are not. Hence, neither A_5 nor A_6 is isomorphic to $A_4(a)$.

(3) Assume that A_5 and A_6 are isomorphic and there is a nonsingular matrix $X = (x_{ij})$ satisfying

$$ {}^tXT(1,0,0,0,0)X = d \cdot T(1,1,1,0,0) \tag{21}$$

where $d = |X| \neq 0$. Let $Y = (y_{ij})$ be the matrix in the left hand side of (21). We thus have

$$\begin{cases} y_{12} = x_{11}x_{22} + x_{31}x_{32} = d \\ y_{13} = x_{11}x_{23} + x_{31}x_{33} = d \\ y_{22} = x_{12}x_{22} + x_{32}^2 = 0 \\ y_{23} = x_{12}x_{23} + x_{32}x_{33} = d. \end{cases} \tag{22}$$

On the other hand, by (10) we have

$$\mathrm{jac}(A_5) = (0,0,1)T(1,0,0,0,0)\,{}^t(e_1,e_2,e_3) = e_3$$

and

$$\mathrm{jac}(A_6) = (1,-1,1)T(1,1,1,0,0)\,{}^t(e_1,e_2,e_3) = e_2 + e_3.$$

Hence by Proposition 3.4 we have $(0,0,1)X = (0,d,d)$, and so $x_{31} = 0$ and $x_{32} = x_{33}$. Hence, by (22)

$$y_{12} = x_{11}x_{22} = y_{13} = x_{11}x_{23} = d \neq 0.$$

It follows that $x_{22} = x_{23}$. However, by (22) we get a contradiction

$$d = y_{23} - y_{22} = x_{12}(x_{23} - x_{22}) + x_{32}(x_{33} - x_{32}) = 0.$$

□

As mentioned in the introduction, let us define the equivalence relation $\sim$ in K by $a \sim b$ if and only if $a = \pm b$ for $a, b \in K$. By (1) of Lemma 6.2 we have $A_4(a) \cong A_4(b)$ if and only if $a \sim b$. Choose and fix a complete set $\mathcal{H} \subset K$ of equivalence class representatives for $K/\sim$. This $\mathcal{H}$ will appear in the next two sections too. By the above arguments (i) $\sim$ (iv) and Lemma 6.2, we have

Proposition 6.3. *Up to isomorphism, straight algebras of rank 2 are classified into three families*

$$A_4(a) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & a \\ 0 & 0 & 1 \end{pmatrix} \ (a \in \mathcal{H}),\ A_5 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } A_6 = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

7. Straight algebras of rank 3

Let A be a straight algebra of rank 3. Then, by Corollary 5.3, we may suppose that A is defined by

$$A(a,b,c) = T(a,b,c,1,1) = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$$

with $a,b,c \in K$. For $x \in K$, let $A_7(x) = A(x,0,0)$. We have ${}^tUA(0,0,c)U = A_7(c)$ with the transformation matrix $U = \begin{pmatrix} 0 & 0 & 1 \\ -c & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$. Hence, A is isomorphic to $A_7(c)$, if $a = b = 0$. Below, we assume that $(a,b) \neq (0,0)$ and let

$$h = \sqrt{a^2 + b^2 - abc} \quad \text{and} \quad d = \sqrt{a^2 + b^2 + c^2 - abc}.$$

(I) The case where $h \neq 0$ and $d \neq 0$. Put

$$X = \begin{pmatrix} 0 & h/d & c/d \\ -a/h & (bc - ad^2)/hd & -b/d \\ (ac-b)/h & ((ac-b)d^2 - ac)/hd & a/d \end{pmatrix}.$$

Then we have

$${}^tXA(a,b,c)X = A_7(d),$$

hence A is isomorphic to $A_7(d)$.

(II) The case where $h = 0, d \neq 0$ and $a^2 \neq c^2$. In this case, $a \neq 0$, $b \neq 0$ and $c = (a^2+b^2)/ab$. Let

$$Y_1 = \begin{pmatrix} (b-ac)/f & -a/f & 1 \\ c/f & ab/cf & -b/c \\ 0 & f/c & a/c \end{pmatrix},$$

where $f = \sqrt{c^2 - a^2}$. Then we have

$${}^tY_1A(a,b,c)Y_1 = A_7(c),$$

hence A is isomorphic to $A_7(c)$.

(III) The case where $h = 0, d \neq 0, a^2 = c^2$ and $a^2 \neq 4$. In this case, $c = \sigma a$ and $b = a(\sigma a + g)/2$, where $g = \tau\sqrt{a^2 - 4}$ and $\sigma, \tau \in \{1, -1\}$. Let

$$Y_2 = \begin{pmatrix} (a - \sigma g)/2g & 1/g & \sigma \\ (2 - \sigma ag - a^2)/2g & -(a + 3\sigma g)/2g & -(\sigma a + g)/2 \\ 1 & (a - \sigma g)/2 & 1 \end{pmatrix}.$$

Then we have

$${}^tY_2A(a,b,c)Y_2 = A_7(a),$$

hence A is isomorphic to $A_7(a)$.

(IV) The case where $h = 0, d \neq 0, a^2 = c^2$ and $a^2 = 4$. In this case, we have

$$(a, b, c) = (2, 2, 2), (2, -2, -2), (-2, -2, 2) \text{ or } (-2, 2, -2).$$

Let $A_8 = A(2, 2, 2)$, and then $A(2, -2, -2), A(-2, -2, 2)$ and $A(-2, -2, -2)$ are isomorphic to A_8 with the transformation matrices

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -2 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & -1 \\ 0 & -1 & 0 \end{pmatrix},$$

respectively[*3].

(V) The case where $d = 0$ and $a \neq 0$. Let $i = \sqrt{-1}$ and $A_9 = A(1, i, 0)$. First, $A(a, b, c)$ and $A(a, ia, 0)$ are isomorphic with the transformation matrix

$$\begin{pmatrix} i(ac - b)/a & ic/a & 0 \\ -ic/a & -ib/a & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Secondly, $A(a, ia, 0)$ and A_9 are isomorphic with the transformation matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & (a^2 + 1)/2a & -i(a^2 - 1)/2a \\ 0 & i(a^2 - 1)/2a & (a^2 + 1)/2a \end{pmatrix}.$$

Consequently, A is isomorphic to A_9.

(VI) The case where $d = 0, h \neq 0$ and $a = 0$. In this case, we have $b \neq 0$ and $c = i\sigma b$, where $\sigma \in \{1, -1\}$. Put

$$Z = \begin{pmatrix} b & (b^2 + 1)/2b & -i(b^2 - 1)/2b \\ i\sigma b & i\sigma(b^2 - 1)/2b & \sigma(b^2 + 1)/2b \\ -1 & 0 & 0 \end{pmatrix}.$$

Then we have

$${}^t Z A(a, b, c) Z = A_9,$$

hence A is isomorphic to A_9.

(VII) The case where $d = 0, h = 0$ and $a = 0$. This case does not occur because $(a, b) \neq (0, 0)$.

Lemma 7.1.

(1) *For $a, b \in K$, $A_7(a) \cong A_7(b)$ if and only if $a = \pm b$.*
(2) *For all $a \in K$, $A_8 \not\cong A_7(a)$ and $A_9 \not\cong A_7(a)$.*
(3) *$A_8 \not\cong A_9$.*

[*3]This case is the only one where there are only a finite number of isomorphic algebras of the triangular form $A(a, b, c)$. Moreover, as we have observed, there were no such cases for $B(a, b)$ nor $C(a, b, c)$. In this sense, A_8 can be called a *sporadic* zeropotent algebra.

Proof. (1) and (2). First, $A_7(-a)$ is isomorphic to $A_7(a)$ with the transformation matrix $\begin{pmatrix} 0 & 1 & 0 \\ 1 & a & 0 \\ 0 & 0 & 1 \end{pmatrix}$, and if $a \neq 0$, then $A_7(0)$ is never isomorphic to $A_7(a), A_8$ and A_9 because $A_7(0)$ is symmetric but $A_7(a), A_8$ and A_9 are not. We thus may assume $a \neq 0$.

Next, suppose that $A = A_7(d)$ is isomorphic to $A' = A(a, b, c)$, where $b, c \in K$ and $a, d \in K \setminus \{0\}$. Because $|A| = |A'| = 1$, by Corollary 2.3 there is a matrix $X = (x_{ij})$ such that $|X| = 1$ and

$$ {}^tXAX = A'. \tag{23} $$

By (10) we have

$$ \begin{cases} \mathrm{jac}(A) = (0,0,d)A_7(d)\,{}^t(e_1, e_2, e_3) = de_3 \\ \mathrm{jac}(A') = (c, -b, a)A'\,{}^t(e_1, e_2, e_3) = ce_1 + (ac - b)e_2 + ae_3. \end{cases} \tag{24} $$

Hence, by Proposition 3.4, $(0, 0, d)X = (c, ac - b, a)$, that is,

$$ dx_{31} = c, dx_{32} = ac - b \text{ and } dx_{33} = a. \tag{25} $$

Now, (23) is expanded as

$$ \begin{cases} p_1 = x_{11}^2 + dx_{11}x_{21} + x_{21}^2 + x_{31}^2 - 1 = 0 \\ p_2 = x_{11}x_{12} + dx_{11}x_{22} + x_{21}x_{22} + x_{31}x_{32} - a = 0 \\ p_3 = x_{11}x_{13} + dx_{11}x_{23} + x_{21}x_{23} + x_{31}x_{33} - b = 0 \\ p_4 = x_{11}x_{12} + dx_{12}x_{21} + x_{21}x_{22} + x_{31}x_{32} = 0 \\ p_5 = x_{12}^2 + dx_{12}x_{22} + x_{22}^2 + x_{32}^2 - 1 = 0 \\ p_6 = x_{12}x_{13} + dx_{12}x_{23} + x_{22}x_{23} + x_{32}x_{33} - c = 0 \\ p_7 = x_{11}x_{13} + dx_{13}x_{21} + x_{21}x_{23} + x_{31}x_{33} = 0 \\ p_8 = x_{12}x_{13} + dx_{13}x_{22} + x_{22}x_{23} + x_{32}x_{33} = 0 \\ p_9 = x_{13}^2 + dx_{13}x_{23} + x_{23}^2 + x_{33}^2 - 1 = 0. \end{cases} $$

Then we have

$$ p_2 - p_4 = d(x_{11}x_{22} - x_{12}x_{21}) - a = 0, \tag{26} $$

$$ p_6 - p_8 = d(x_{12}x_{23} - x_{13}x_{22}) - c = 0, \tag{27} $$

and

$$ p_3 - p_7 = d(x_{11}x_{23} - x_{13}x_{21}) - b = 0. \tag{28} $$

Let $X_1 = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$, $X_2 = \begin{pmatrix} x_{12} & x_{13} \\ x_{22} & x_{23} \end{pmatrix}$ and $X_3 = \begin{pmatrix} x_{11} & x_{13} \\ x_{21} & x_{23} \end{pmatrix}$, then by (26), (27) and (28) we have

$$ |X_1| = a/d \neq 0, |X_2| = c/d \text{ and } |X_3| = b/d. \tag{29} $$

(i) Suppose that $b = c = 0$. Then by (25), $x_{31} = x_{32} = 0$ and $x_{33} = a/d$. Hence

$$ 1 = |X| = \frac{a}{d}|X_1| = \frac{a^2}{d^2}, $$

and so we have $a = \pm d$. Therefore, by taking $d = b$, we are done with (1) of the lemma.

(ii) Suppose that $a = b = c = 2$. Then by (25), $x_{31} = x_{32} = x_{33} = 2/d$. Hence, by (29) we have

$$|X| = \frac{2}{d}(|X_2| - |X_3| + |X_1|) = \frac{4}{d^2} = 1.$$

It follows that $d = 2\sigma$ with $\sigma \in \{1, -1\}$, and so $x_{31} = x_{32} = x_{33} = \sigma$. Therefore,

$$p_1 = (x_{11} + \sigma x_{21})^2 = p_5 = (x_{12} + \sigma x_{22})^2 = 0.$$

Hence, $x_{21} = -\sigma x_{11}$ and $x_{22} = -\sigma x_{12}$. Consequently, $|X_1| = 0$, contradicting (29). Therefore we see that $A_7(a)$ and A_8 are not isomorphic.

(iii) Suppose that $a = 1, b = i$ and $c = 0$. By (25) we have $x_{31} = 0, x_{32} = -i/d$ and $x_{33} = 1/d$. Hence, we have

$$|X| = \frac{1}{d}(|X_1| + i|X_3|) = \frac{1}{d}\left(\frac{1}{d} + \frac{i^2}{d}\right) = 0$$

by (29), a contradiction. Thus, $A_7(a)$ and A_9 are not isomorphic.

(3) Assume that A_9 is isomorphic to A_8. Then there is a matrix $X = (x_{ij})$ satisfying $|X| = 1$ and

$$^tXA(1, i, 0)X = A(2, 2, 2),$$

which is expanded as

$$(30)\quad \begin{cases} q_1 = x_{11}^2 + x_{21}(x_{11} + x_{21}) + x_{31}(ix_{11} + x_{31}) - 1 = 0 \\ q_2 = x_{11}x_{12} + x_{22}(x_{11} + x_{21}) + x_{32}(ix_{11} + x_{31}) - 2 = 0 \\ q_3 = x_{11}x_{13} + x_{23}(x_{11} + x_{21}) + x_{33}(ix_{11} + x_{31}) - 2 = 0 \\ q_4 = x_{11}x_{12} + x_{21}(x_{12} + x_{22}) + x_{31}(ix_{12} + x_{32}) = 0 \\ q_5 = x_{12}^2 + x_{22}(x_{12} + x_{22}) + x_{32}(ix_{12} + x_{32}) - 1 = 0 \\ q_6 = x_{12}x_{13} + x_{23}(x_{12} + x_{22}) + x_{33}(ix_{12} + x_{32}) - 2 = 0 \\ q_7 = x_{11}x_{13} + x_{21}(x_{13} + x_{23}) + x_{31}(ix_{13} + x_{33}) = 0 \\ q_8 = x_{12}x_{13} + x_{22}(x_{13} + x_{23}) + x_{32}(ix_{13} + x_{33}) = 0 \\ q_9 = x_{13}^2 + x_{23}(x_{13} + x_{23}) + x_{33}(ix_{13} + x_{33}) - 1 = 0. \end{cases}$$

Because $\text{jac}(A_9) = -ie_2 + e_3$ and $\text{jac}(A_8) = 2(e_1 + e_2 + e_3)$ by the second equation in (24), we have $(0, -i, 1)X = (2, 2, 2)$ by Proposition 3.4, that is,

$$(31)\quad -ix_{21} + x_{31} = -ix_{22} + x_{32} = -ix_{23} + x_{33} = 2.$$

Hence, by (30) and (31), we have

$$\begin{cases} q_2 = x_{11}x_{12} + 2i(x_{11} + x_{21} + x_{22} - i) = 0 \\ q_3 = x_{11}x_{13} + 2i(x_{11} + x_{21} + x_{23} - i) = 0 \\ q_4 = x_{11}x_{12} + 2i(x_{12} + x_{21} + x_{22} - 2i) = 0 \\ q_6 = x_{12}x_{13} + 2i(x_{12} + x_{22} + x_{23} - i) = 0 \\ q_7 = x_{11}x_{13} + 2i(x_{13} + x_{21} + x_{23} - 2i) = 0 \\ q_8 = x_{12}x_{13} + 2i(x_{13} + x_{22} + x_{23} - 2i) = 0. \end{cases}$$

Thus, we reach a contradiction

$$0 = (q_3 + q_4 + q_8) - (q_2 + q_6 + q_7) = 2.$$

□

By the above arguments (I)∼(VII) together with Lemma 7.1, we obtain

Proposition 7.2. *Up to isomorphism, straight algebras of rank 3 are classified into three families*

$$A_7(a) = \begin{pmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \ (a \in \mathcal{H}),\ A_8 = \begin{pmatrix} 1 & 2 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } A_9 = \begin{pmatrix} 1 & 1 & i \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

8. Summary and Concluding Remarks

By summarizing Propositions 4.1, 6.1, 6.3 and 7.2, the following theorem is obtained.

Theorem 8.1. *Up to isomorphism, three-dimensional zeropotent algebras over an algebraically closed field of characteristic not equal to two are classified into the algebras*

$$A_0, A_1, A_2, A_3, \{A_4(a)\}_{a\in\mathcal{H}}, A_5, A_6, \{A_7(a)\}_{a\in\mathcal{H}}, A_8 \text{ and } A_9$$

defined by

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & a \\ 0 & 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 1 & i \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

respectively.

We remark that as easily seen from the proofs, the assumption that K is algebraically closed can be replaced by the weaker one that K is *square-rootable*, that is, for any $a \in K$ there exists $x \in K$ such that $x^2 = a$.

As a final remark, by Theorem 8.1 and Proposition 3.2, we can obtain a classification of three-dimensional complex Lie algebras as follows. Up to isomorphism, three-dimensional Lie algebras over $\mathbb{C}$ are classified into the algebras

$$A_0, A_1, A_2, A_3, \{A_4(a)\}_{a\in\mathcal{H}} \text{ and } A_7(0),$$

where $\mathcal{H}$ is the half plane $\{z \in \mathbb{C} \mid -\pi/2 < \arg(z) \le \pi/2\} \cup \{0\}$.

While [1] gives the well-known classification of three-dimensional real Lie algebras, [3, 2] present a classification of three-dimensional complex Lie algebras. They determine such a Lie algebra L for each dimension of the derived algebra $L' = [L, L]$, where the dimension corresponds to the rank of A in our setting. Although there is a difference in the way of classification between the two, our classification is in accordance with that of [2] by the following correspondence:

A_0	–	Type 1
A_1	–	Type 2a
A_2	–	Type 2b
A_3	–	Type 3 (L_1)
$A_4(0)$	–	Type 3 ($L_{2,1}$)
$\{A_4(a)\}_{a\in\mathcal{H}\setminus\{0\}}$	–	Type 3 ($L_{3,d}$)
$A_7(0)$	–	Type 4.

Our future work includes a classification of three-dimensional zeropotent algebras over the real number field $\mathbb{R}$ as well as a field of characteristic 2.

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